

Lecture 3

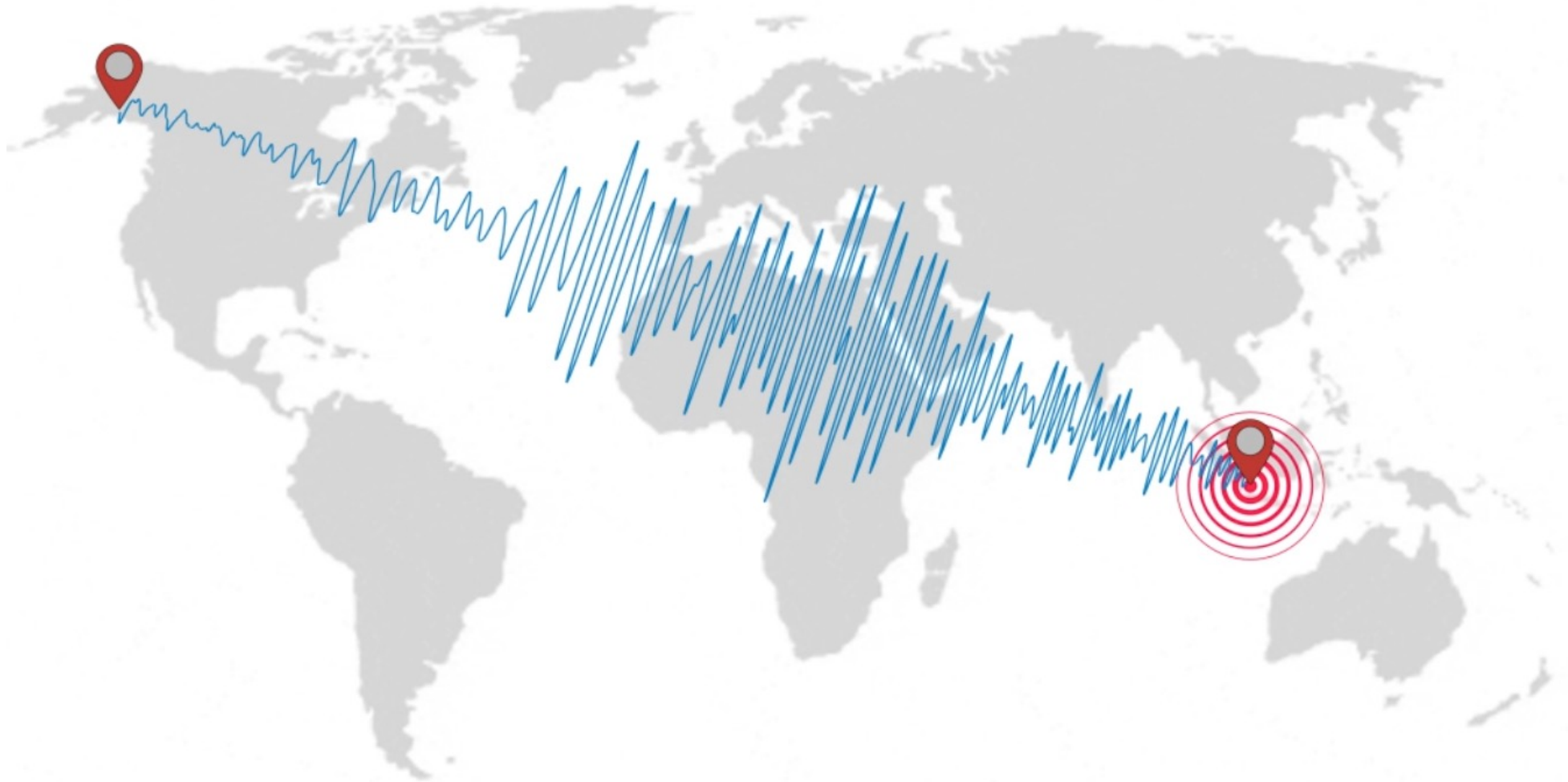
Frequency Domain view of signals

Peter Y K Cheung

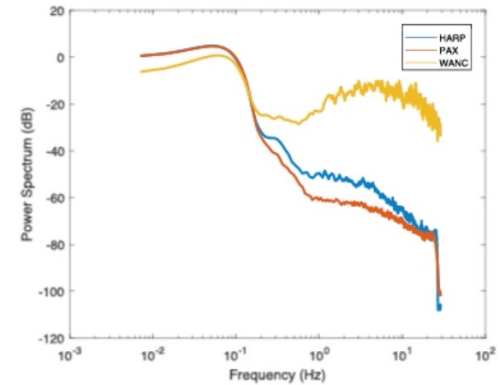
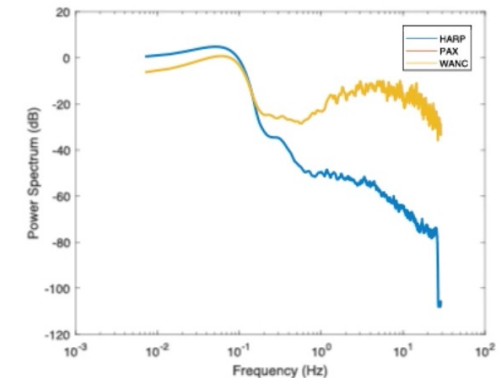
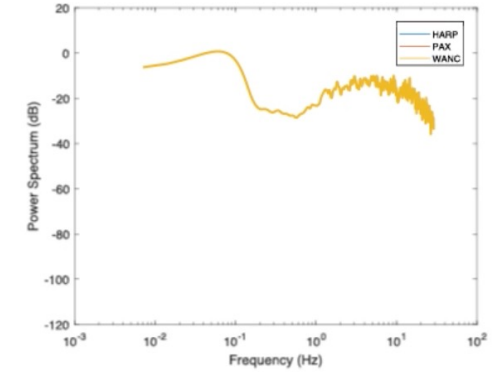
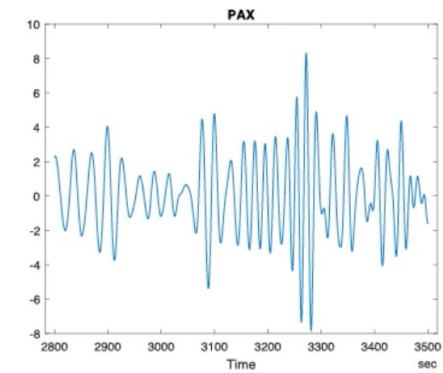
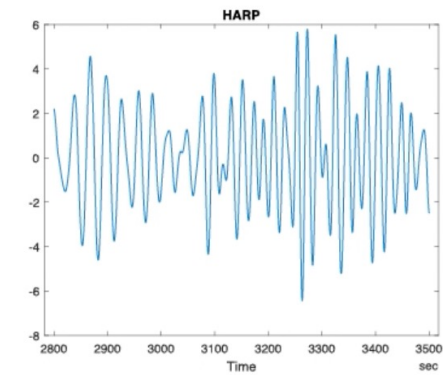
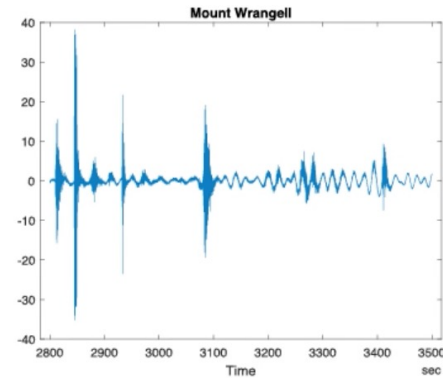
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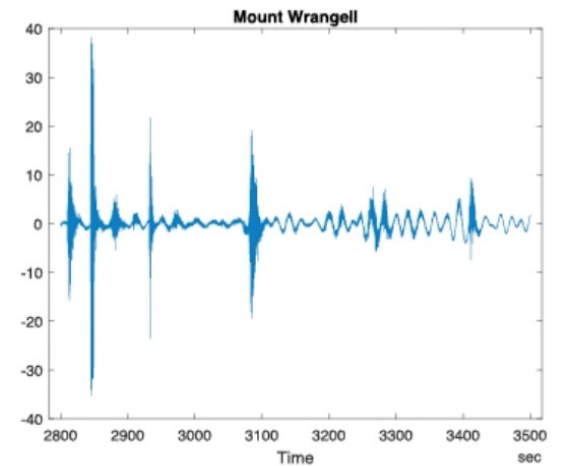
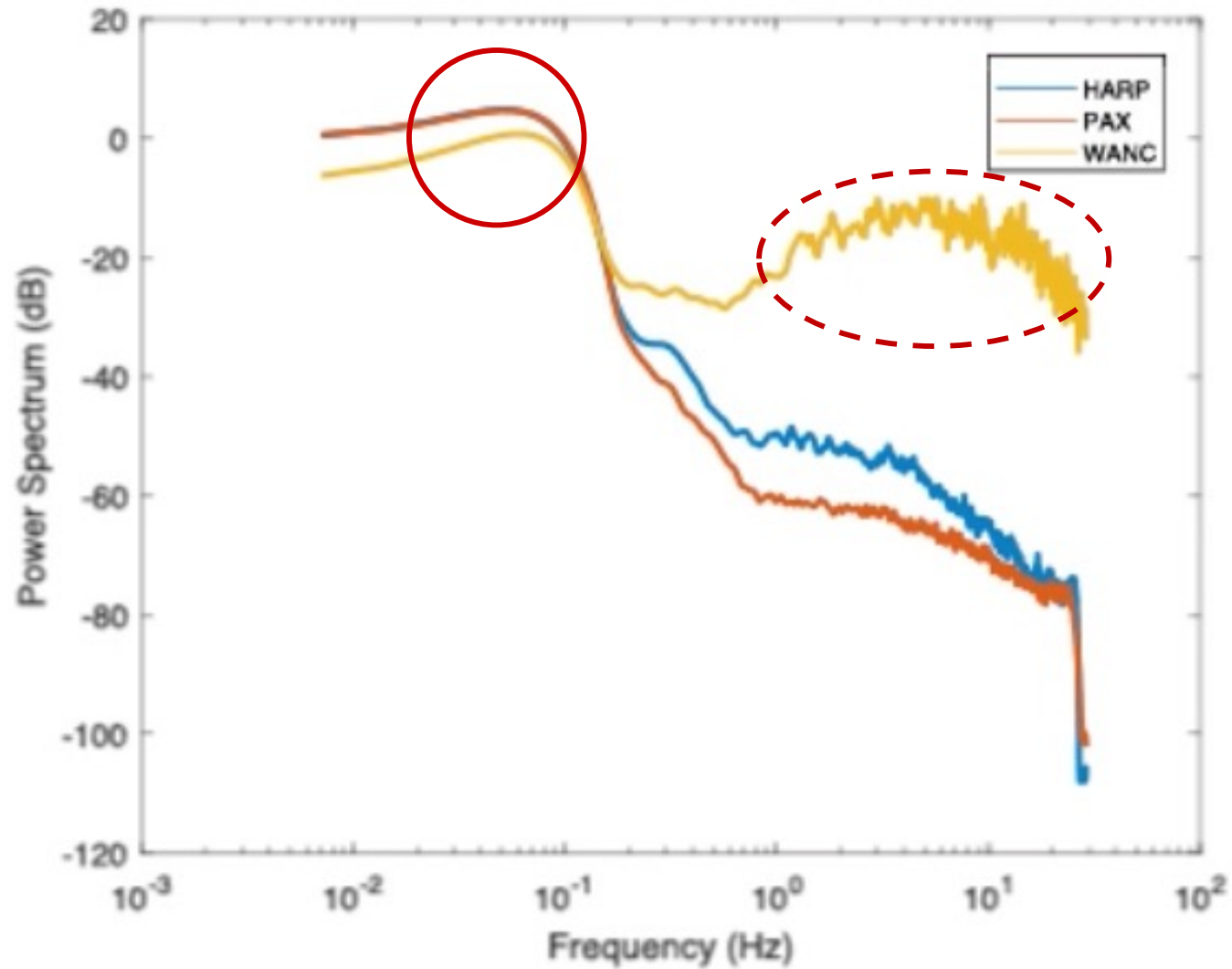
Example – Seismic signal measured in Alaska due to Sumatra Earthquake in 2004



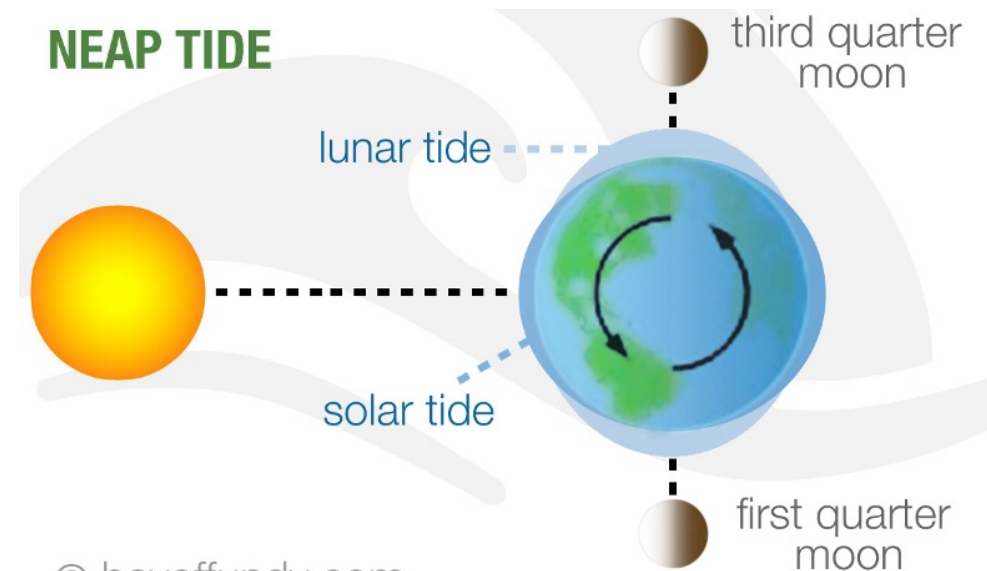
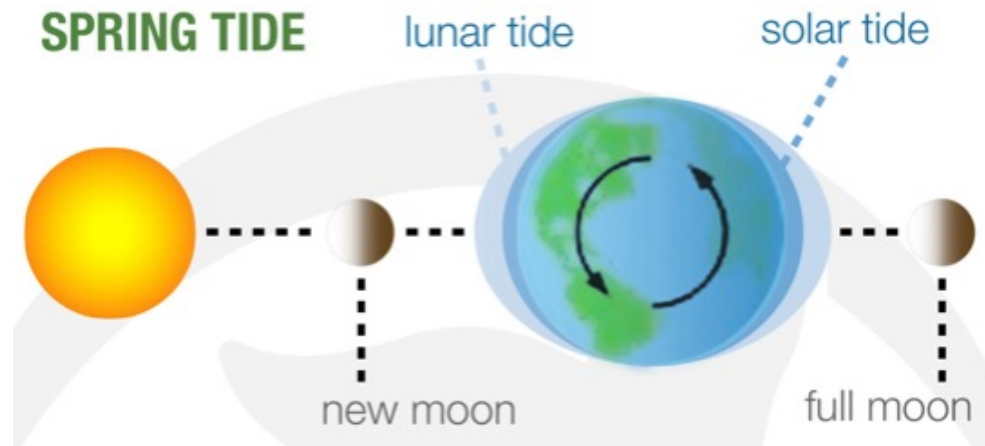
Three stations recorded data for analysis



Frequency domain view of signal – more informative



Prediction of Tides

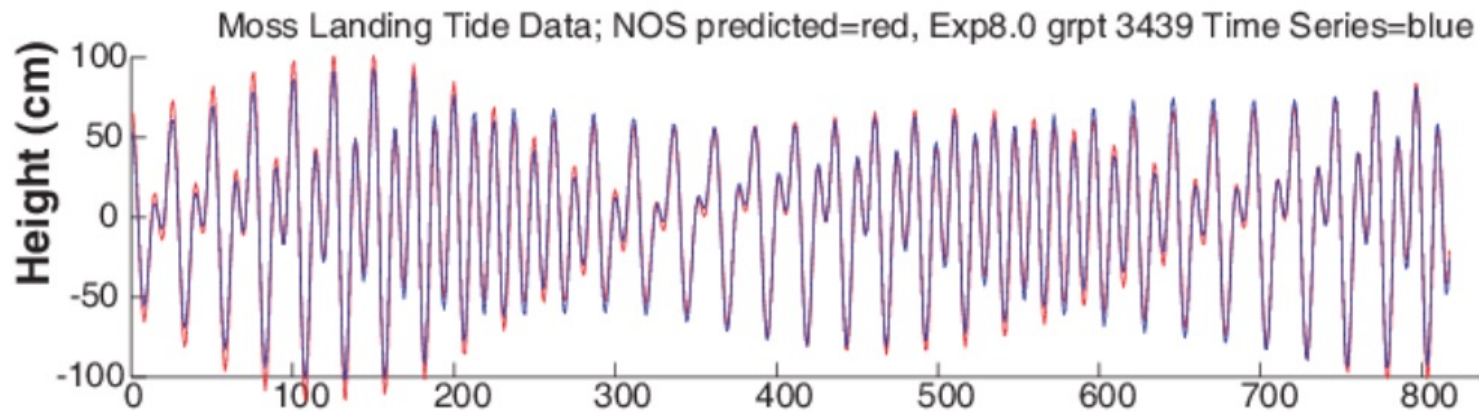
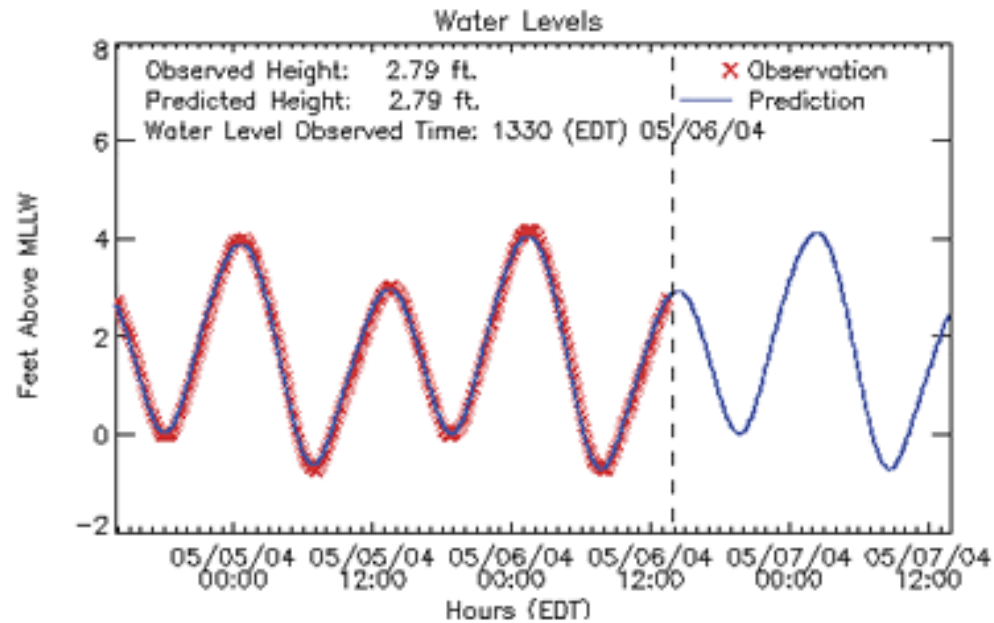


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- ◆ Early scientists:
 - Isaac Newton
 - Joseph Fourier
 - William Thomson (Lord Kelvin)
 - Pierre-Simon Laplace

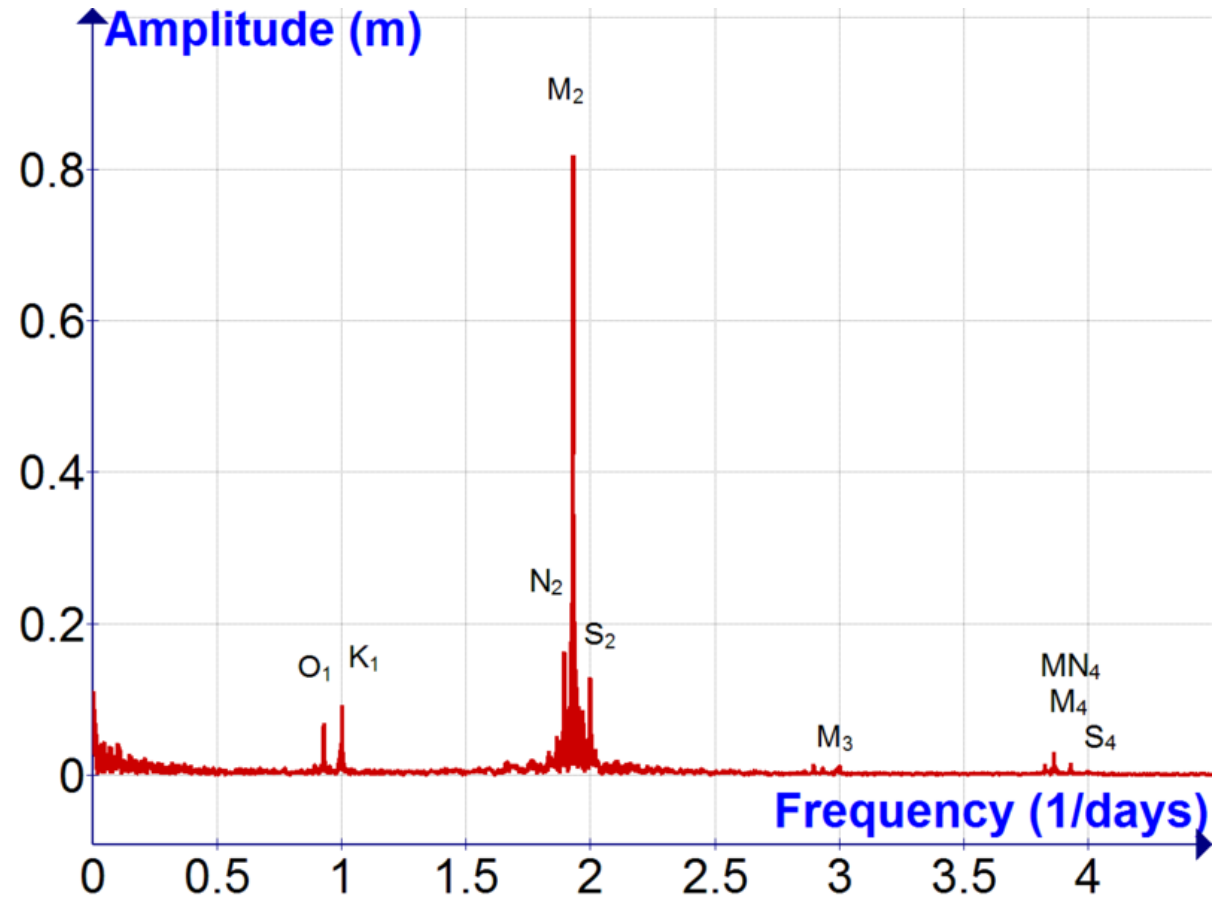
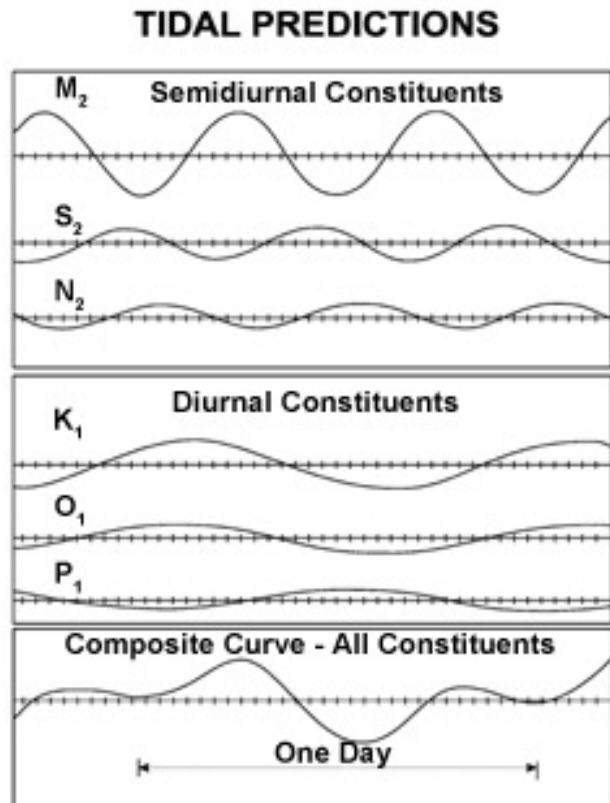
<http://oceanmotion.org/html/background/tides-observing.htm>

Periodicity of the Tides



<http://oceanmotion.org/html/background/tides-observing.htm>

Tides decomposed into periodic constituents



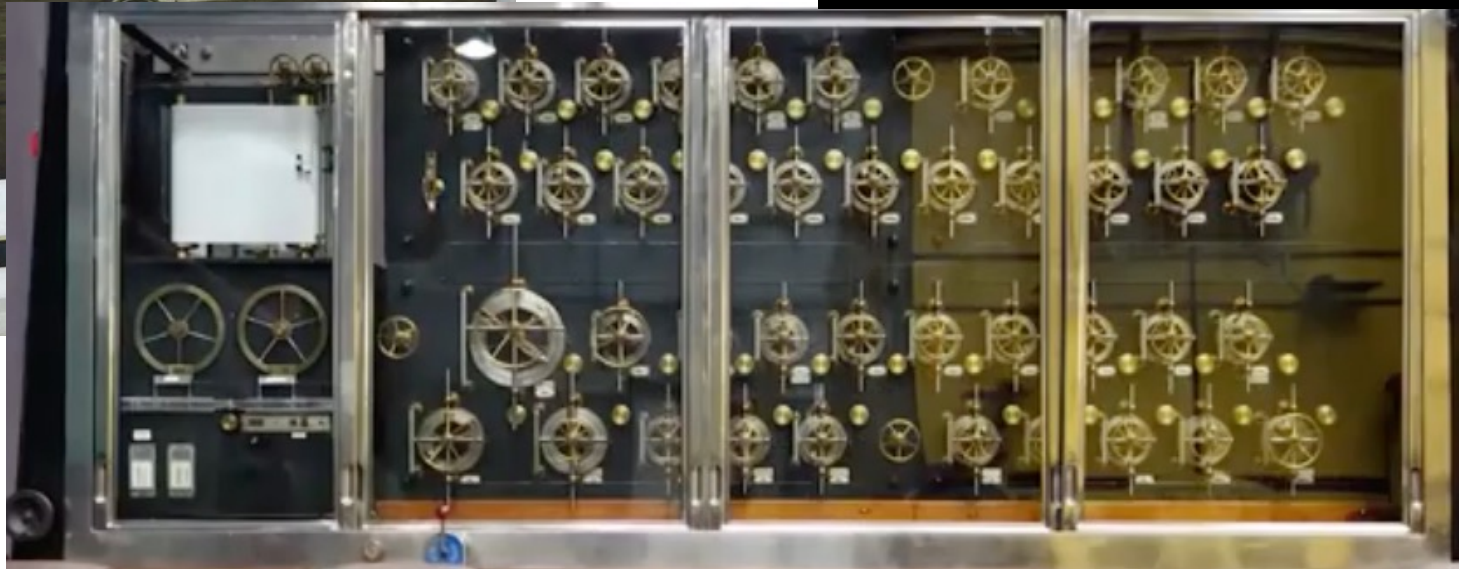
Kelvin's Tide Prediction Machine

First tide prediction computer



William Thomson (Lord Kelvin)

Doodson_lege machine in Liverpool



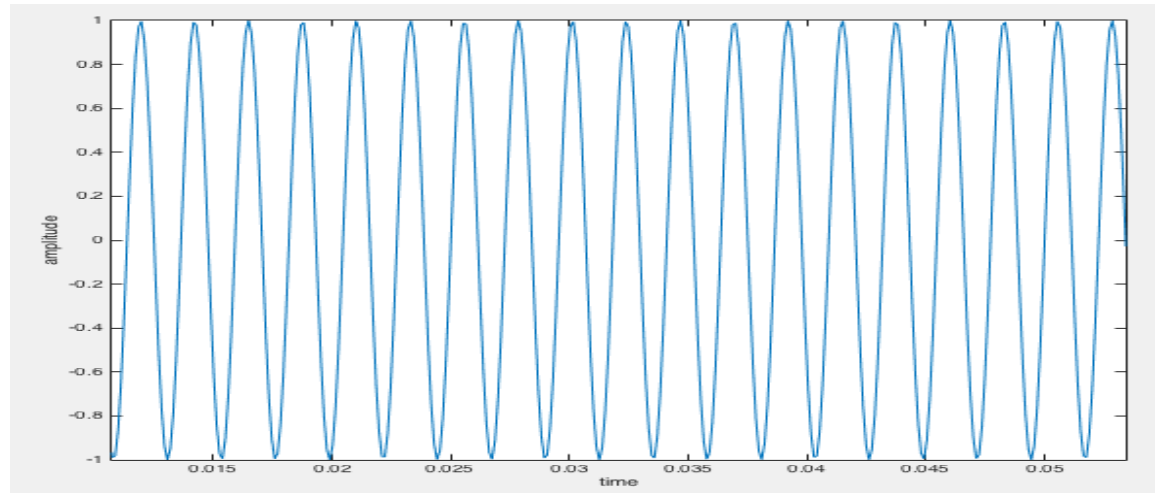
Animation of the Tide Prediction Machine



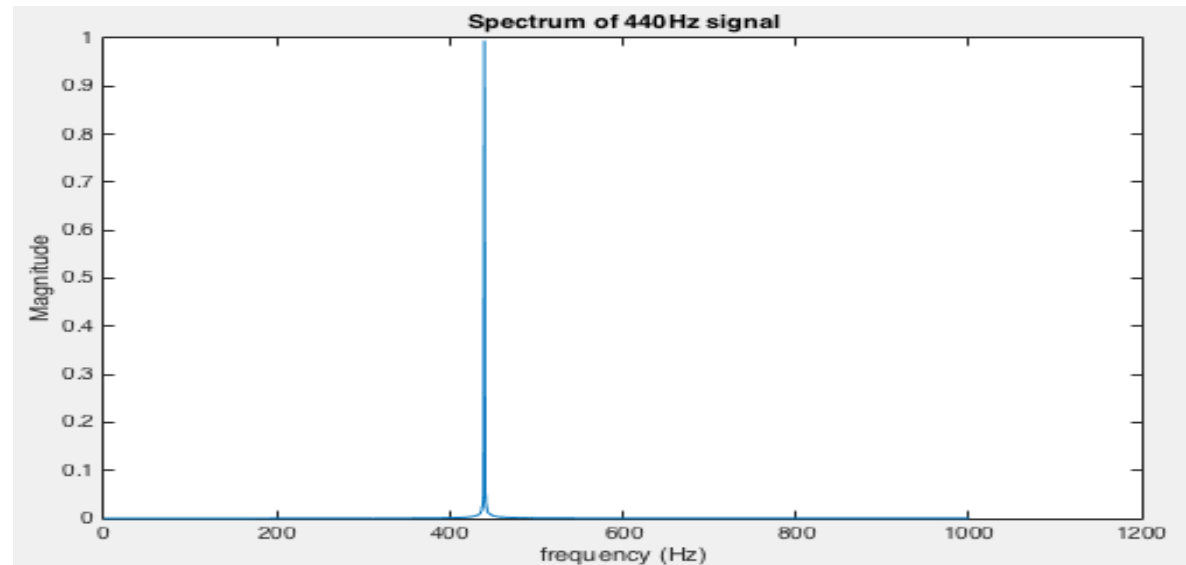
<https://www.ams.org/publicoutreach/feature-column/fcarc-tidesiii3>

Time vs Frequency view of a sinewave

- ◆ Sinewave (sinusoidal signal) in time domain

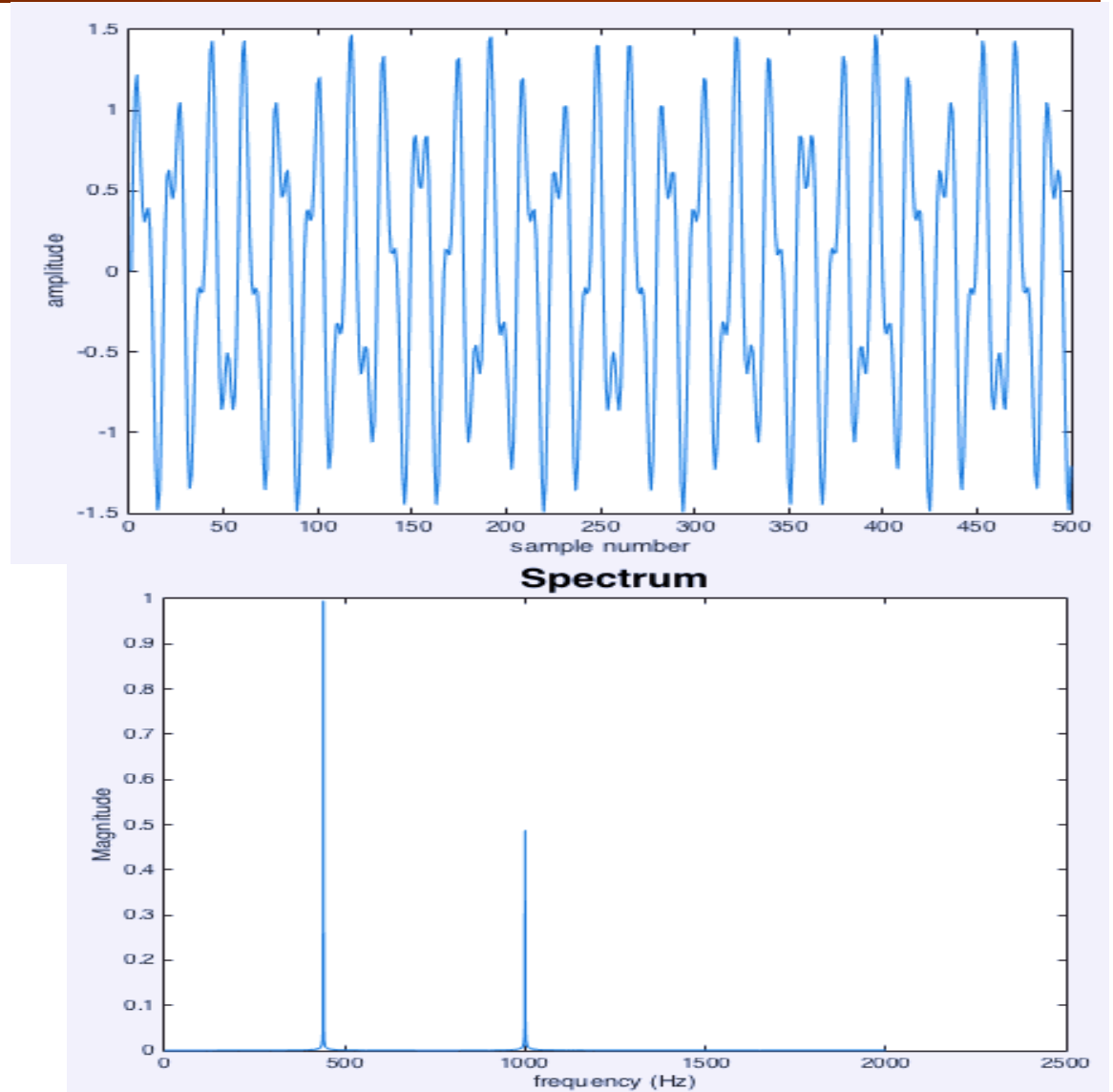


- ◆ Same sinewave in frequency domain



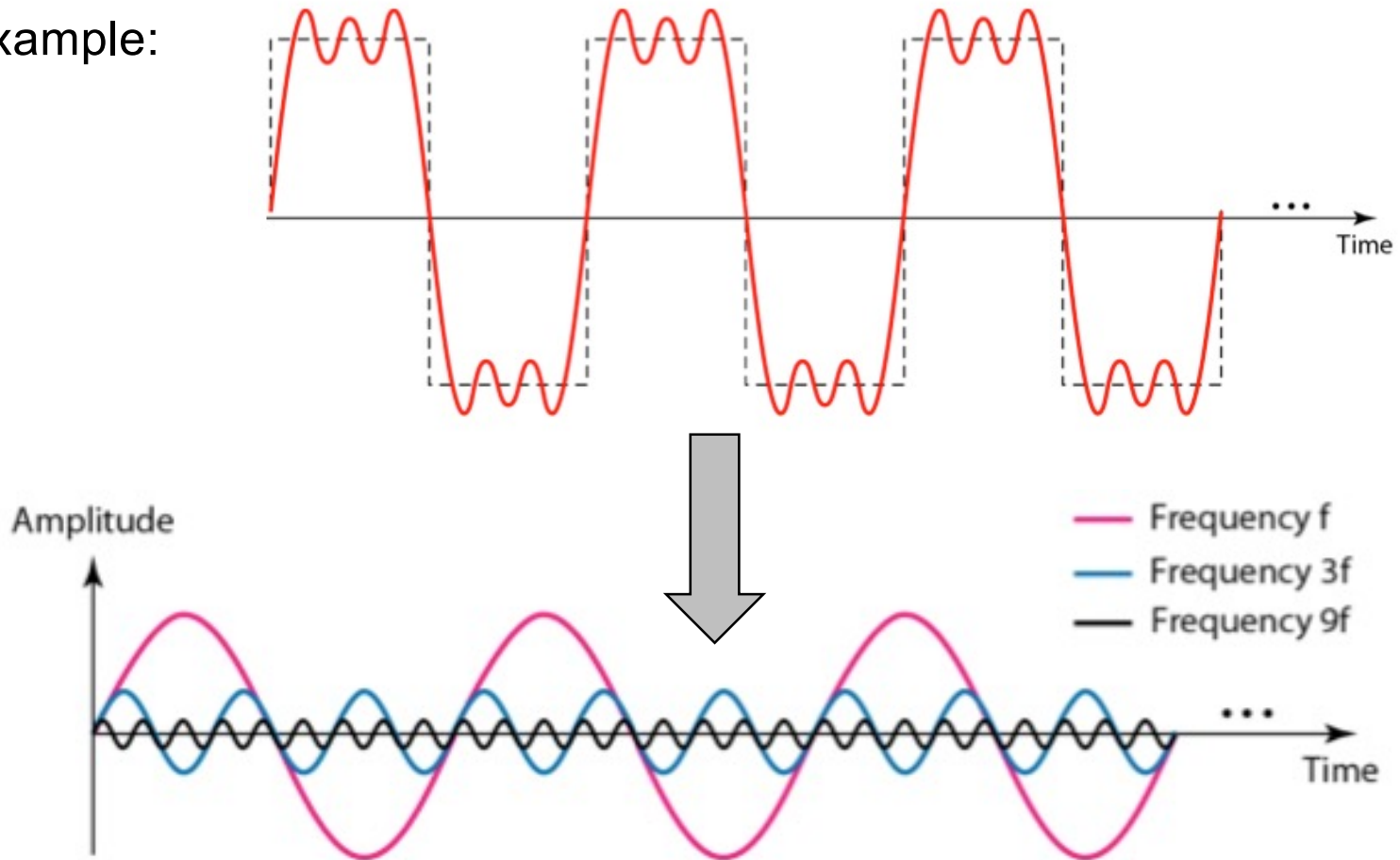
Two sinewaves

- ◆ Adding 440Hz to 1kHz signal. The 440Hz is twice as large as the 1kHz signal.
- ◆ Spectrum of two sinewaves



Key idea – Fourier's theory

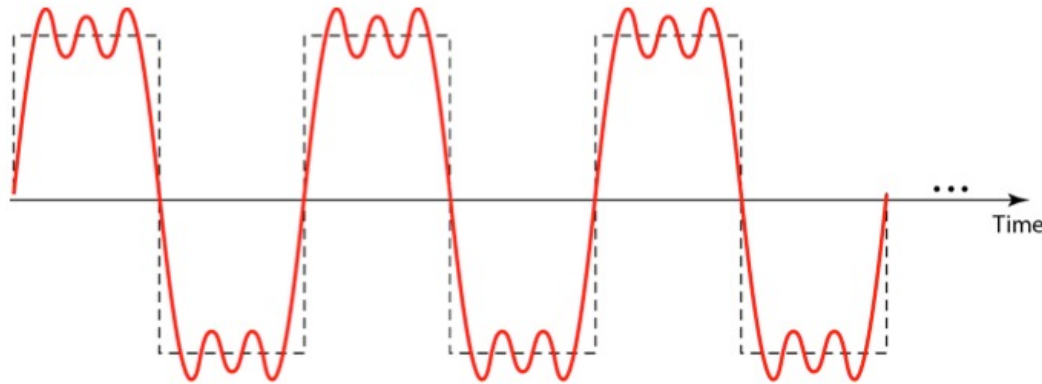
- ◆ Basic idea – any time domain signal can be constructed from **weighted linear sum** of sinusoidal signals (sine or cosine signals) at different frequencies.
- ◆ For example:



Spectrum – Frequency domain representation

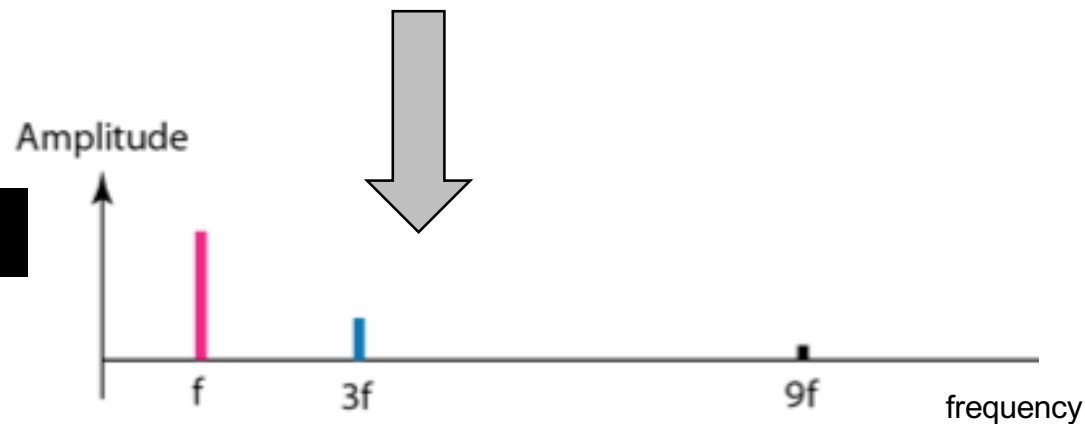
- ◆ Instead of having to store individual time samples, we only need to store the amplitude, frequency and phase of each sinusoidal signal.

Time domain



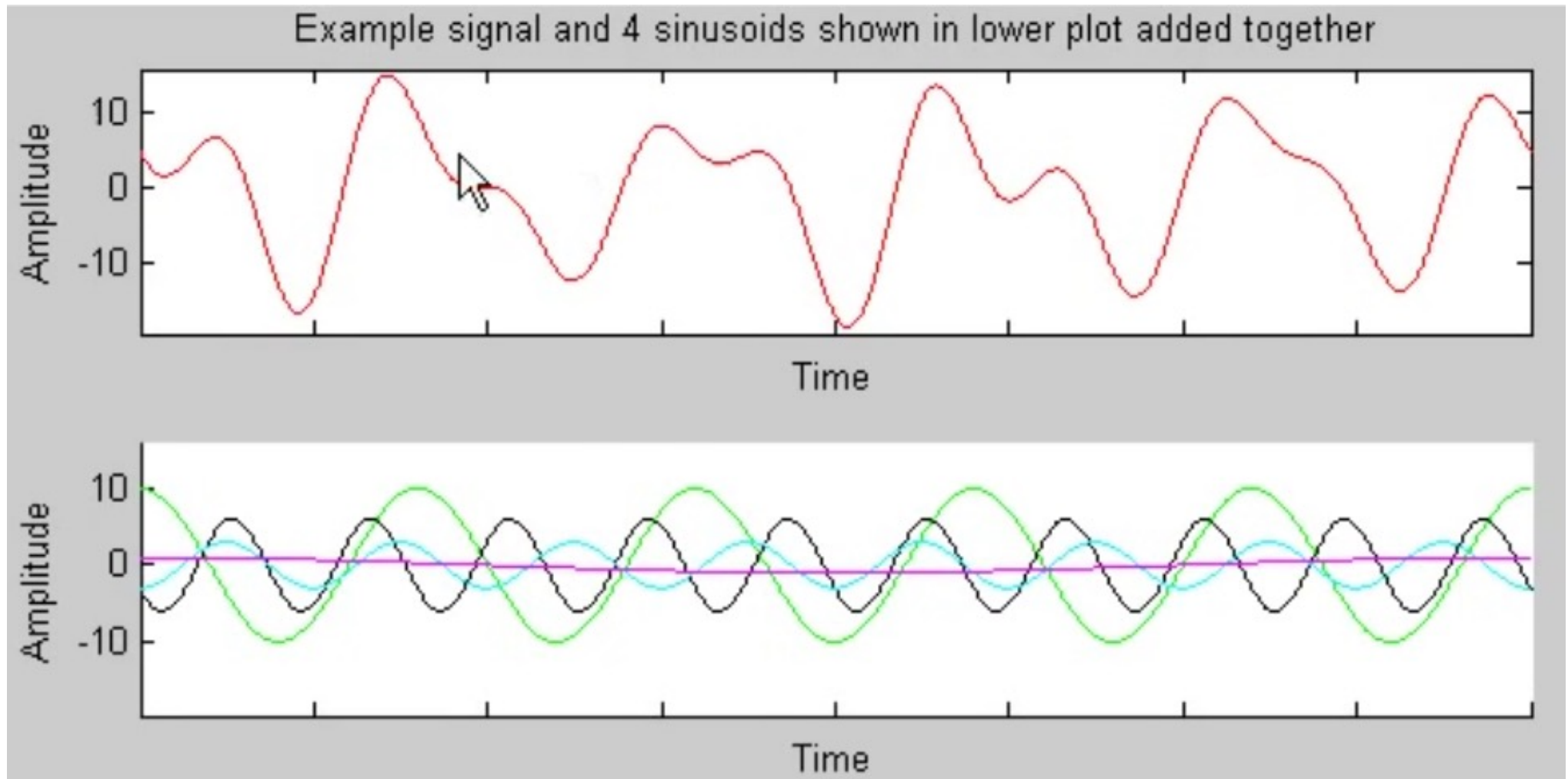
- ◆ Spectrum of signal in frequency domain is represented by amplitude value for each frequency. There is also phase vs frequency, which is not shown here.

Frequency domain



Another Example

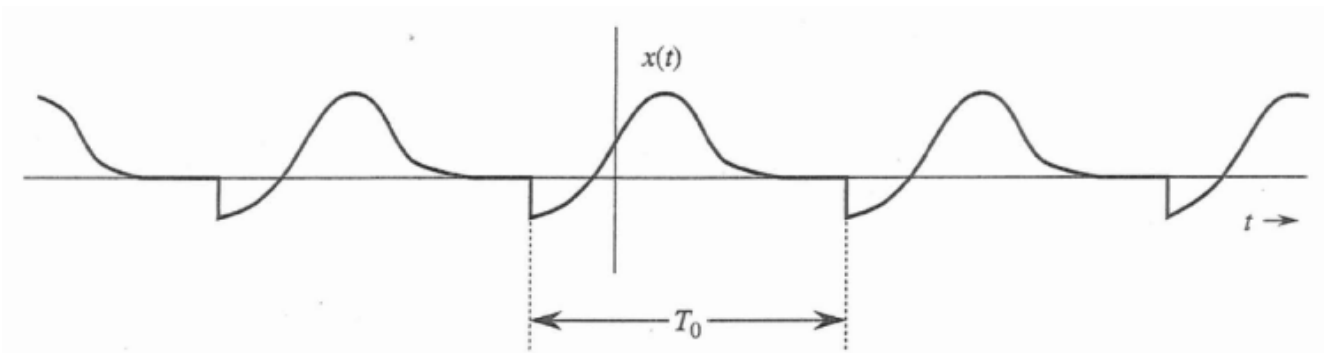
- ◆ Here is another time domain signal that is constructed with four sine waves:



Periodic Signal & Fourier Series

- ◆ A periodic signal $x(t)$ with a period of T_0 has the property:

$$x(t) = x(t + T_0) \quad \text{for all } t$$



- ◆ **Fourier series** expresses $x(t)$ as a **weighted linear sum of sinusoids** (or exponentials) of the fundamental frequency $f_0 = 1/T_0$ and all its harmonics nf_0 where $n = 2, 3, 4, \dots$

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{for all integers } n$$

- ◆ ω_0 is called the **fundamental frequency** such that (f_0 in cycles/sec or Hz, ω_0 in radians/sec)
 $\omega_0 = 2\pi f_0 = 2\pi / T_0$ and $n\omega_0$ are the harmonic frequencies
- ◆ a_0 is the DC (mean) value of $x(t)$ and a_n, b_n are the Fourier coefficients at the frequency $n\omega_0$

How to find a_0 ?

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- ◆ To determine a_0 , we multiply both sides by $\cos m\omega_0 t$ and integrate over T_0 :

$$\begin{aligned} \int_0^{T_0} x(t) dt &= a_0 \int_0^{T_0} dt \\ &+ \sum_{n=1}^{\infty} a_n \int_0^{T_0} \cos n\omega_0 t dt \\ &+ \sum_{n=1}^{\infty} b_n \int_0^{T_0} \sin n\omega_0 t dt \end{aligned}$$

- ◆ 2nd and 3rd terms integrates to zero over one period of time. Therefore only the first term survives:

$$\int_0^{T_0} x(t) dt = a_0 \int_0^{T_0} dt = a_0 T_0$$

- ◆ Therefore

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$$

How to find a_n and b_n coefficients? (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- ◆ To determine a_0 , we simply integrate both sides of the equation over one period T_0 :

$$\int_0^{T_0} x(t) \cos m\omega_0 t dt = a_0 \int_0^{T_0} \cos m\omega_0 t dt$$

$$+ \sum_{n=1}^{\infty} a_n \int_0^{T_0} \cos n\omega_0 t \cos m\omega_0 t dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_0^{T_0} \sin n\omega_0 t \cos m\omega_0 t dt$$

- ◆ But:

$$\int_0^{T_0} \cos m\omega_0 t dt = 0 \quad \text{and} \quad \int_0^{T_0} \cos n\omega_0 t \cos m\omega_0 t dt = 0 \quad \text{if } n \neq m$$

- ◆ When $n = m$,

$$\int_0^{T_0} \cos m\omega_0 t \cos m\omega_0 t dt = \frac{T_0}{2}$$

How to find a_n and b_n coefficients? (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- ◆ Therefore, the ONLY term that survives after multiply by $\cos m\omega_0 t$ and integration is:

$$\int_0^{T_0} x(t) \cos m\omega_0 t dt = a_m \frac{T_0}{2}$$

- ◆ Hence, $a_n = \frac{2}{T_0} \int_0^{T_0} x(t) \cos n\omega_0 t dt$ ($m = n$)

- ◆ Similarly to find b_n multiply $x(t)$ by $\sin m\omega_0 t$ and integration over T_0 :

$$\int_0^{T_0} x(t) \sin m\omega_0 t dt = b_m \frac{T_0}{2}$$

- ◆ Hence, $b_n = \frac{2}{T_0} \int_0^{T_0} x(t) \sin n\omega_0 t dt$

Compact form of Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- ◆ A more compact form of the Fourier Series is derived with the trigonometric identity:

$$\begin{aligned} C \cos(\omega_0 t + \theta) &= C \cos \theta \cos \omega_0 t - C \sin \theta \sin \omega_0 t \\ &= a \cos \omega_0 t + b \sin \omega_0 t \end{aligned}$$

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \\ &= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \end{aligned}$$

where

$$C_0 = a_0$$

DC term

$$C_n = \sqrt{a_n^2 + b_n^2}$$

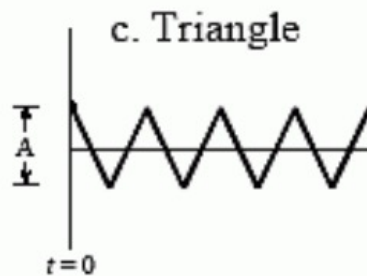
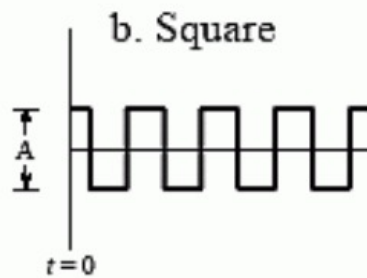
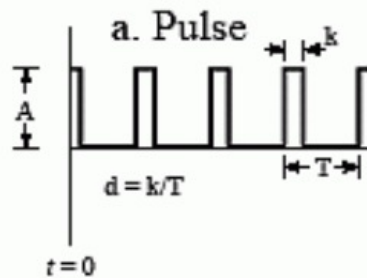
amplitude

$$\theta_n = \tan^{-1} \left(\frac{b_n}{a_n} \right)$$

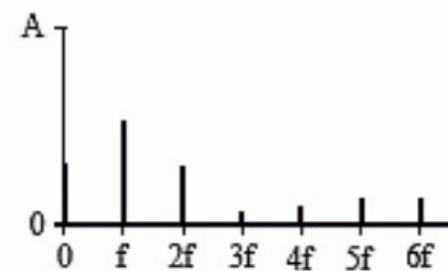
phase angle

Fourier Series of common signals (1)

Time Domain



Frequency Domain

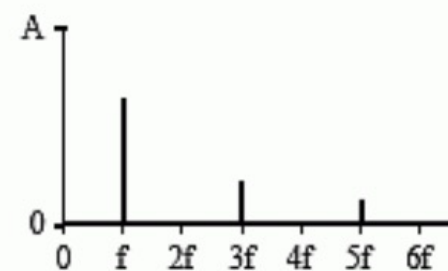


$$a_0 = A d$$

$$a_n = \frac{2A}{n\pi} \sin(n\pi d)$$

$$b_n = 0$$

($d = 0.27$ in this example)

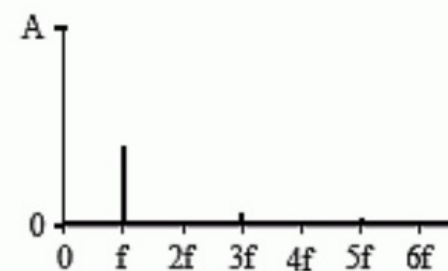


$$a_0 = 0$$

$$a_n = \frac{2A}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$b_n = 0$$

(all even harmonics are zero)



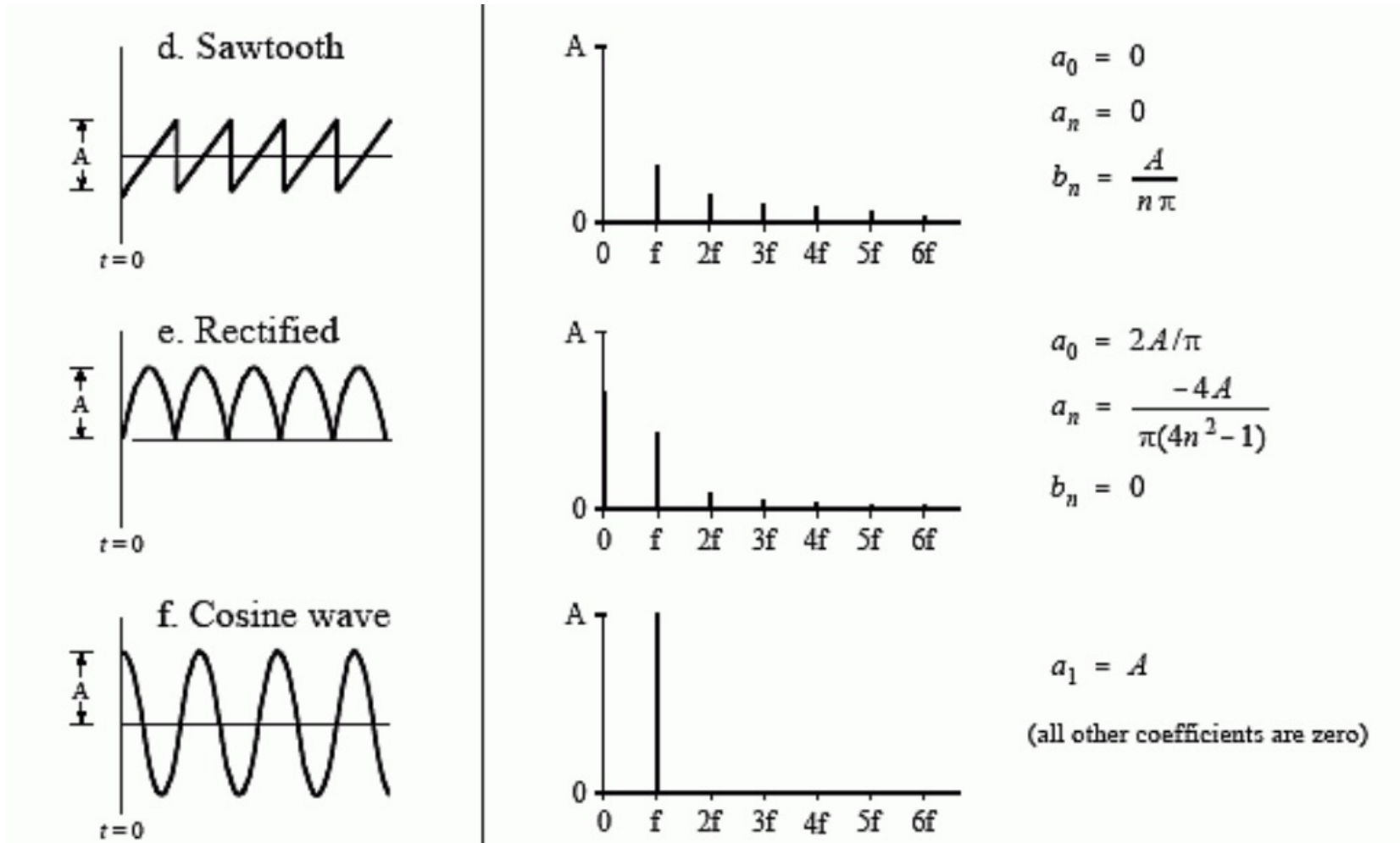
$$a_0 = 0$$

$$a_n = \frac{4A}{(n\pi)^2}$$

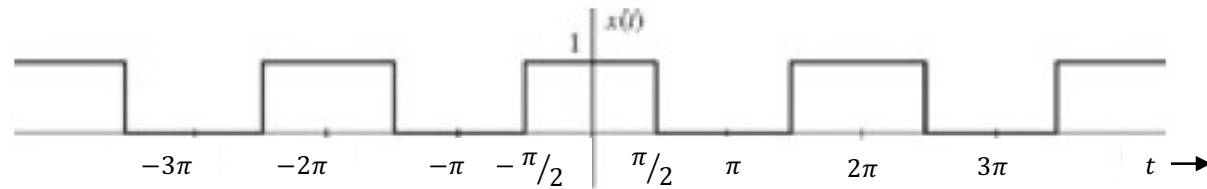
$$b_n = 0$$

(all even harmonics are zero)

Fourier Series of common signals (2)



Fourier series of an even signal

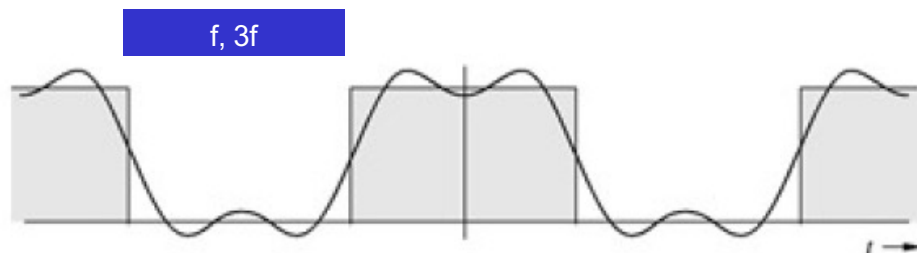
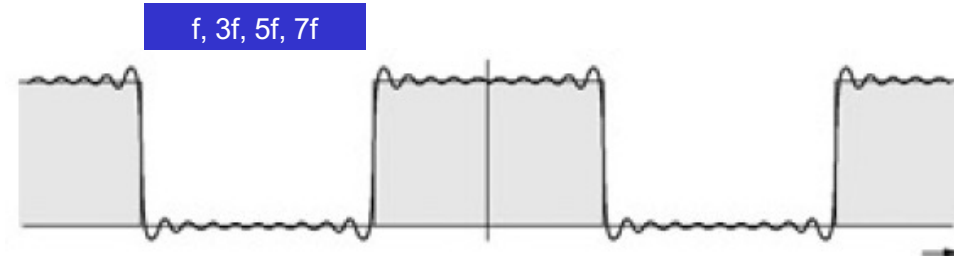
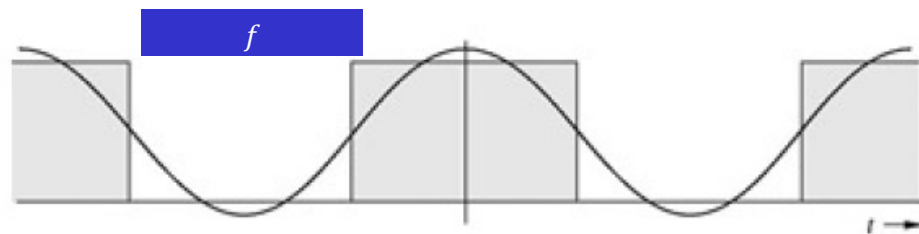
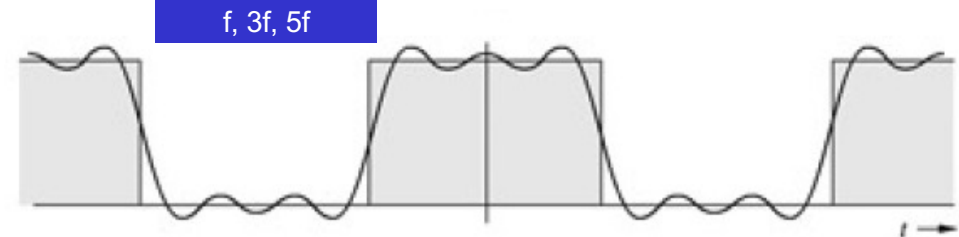
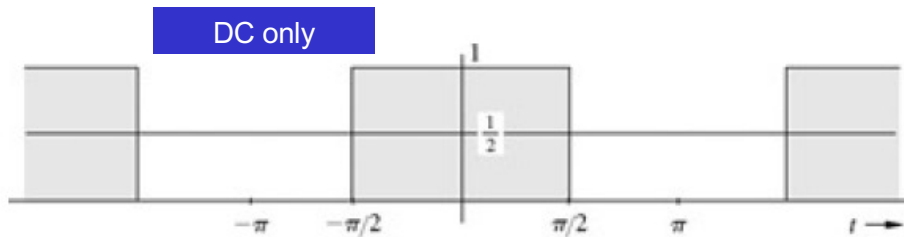
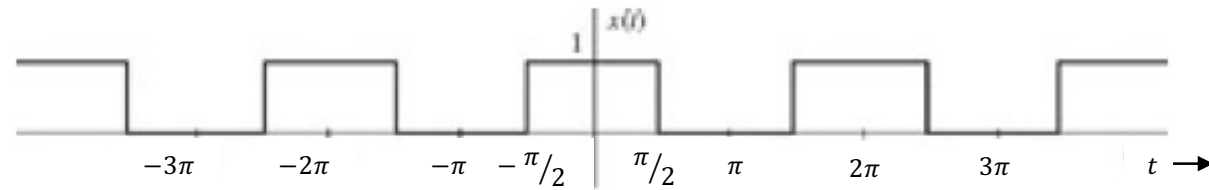


- ◆ The Fourier series for the square-pulse periodic signal shown above is:

$$x(t) = \frac{1}{2} + \frac{2}{\pi} \left(\cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t - \frac{1}{7} \cos 7t + \dots \right)$$

- ◆ The symmetry of this even signal result in three properties:
 1. Such symmetry implies an even even function. Therefore the Fourier series representation only has cosine terms which are also even functions.
 2. This symmetry at $t = 0$ also result in phase angle at all harmonic frequencies = 0.
 3. It only has odd harmonic components – no even harmonic components.

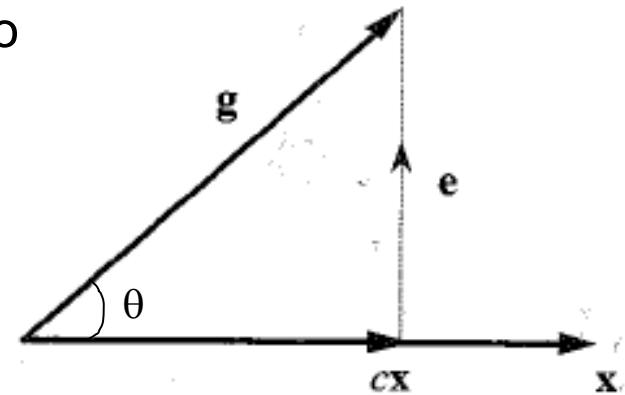
Fourier coefficients and waveshaping



- ◆ Low frequencies determines overall shape
- ◆ High frequencies determines detail structures

A Vector view of Signal

- ◆ To understand why a signal can be represented by linear sum of sinusoidal waveforms, it is useful to consider electrical signals as VECTORS.
- ◆ A vector is specified by its magnitude (or length) and its direction.
- ◆ Consider two vectors \mathbf{g} and \mathbf{x} . If we project \mathbf{g} onto \mathbf{x} , we get $c\mathbf{x}$, where c is a scalar (i.e. constant with no direction).



- ◆ If we approximate \mathbf{g} with $c\mathbf{x}$, then

$$\mathbf{g} = c\mathbf{x} + \mathbf{e}$$

- ◆ \mathbf{e} , the error vector, is minimum when it is perpendicular to \mathbf{x} .
- ◆ $c\mathbf{x}$ is known as the **projection** of \mathbf{g} onto \mathbf{x} .
- ◆ It can be shown (in the notes below) that:

$$c = \frac{\mathbf{g} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{1}{|\mathbf{x}|^2} \mathbf{g} \cdot \mathbf{x}$$

Dot product

$$\mathbf{g} \cdot \mathbf{x} = |\mathbf{g}||\mathbf{x}| \cos \theta$$

Orthogonal Set of signals

- ◆ If vector \mathbf{g} is at right angle to vector \mathbf{x} , then the projection of \mathbf{g} and \mathbf{x} is zero. These two vectors (or signals) are known to be **orthogonal**.
- ◆ It can easily be shown that two sinusoidal signals of DIFFERENT frequencies are orthogonal to each other.
- ◆ The complete set of sinusoidal signals (i.e. of all possible frequency) forms a COMPLETE orthogonal set of signals.
- ◆ What this means is that ALL time domain signals can be formed out of projects (or components) onto these these sinusoidal set of signals!
- ◆ This is the foundation of **Fourier Series** and **Fourier Transform**, which will be discussed further at the next Lecture.

Three Big Ideas

1. **Time domain** view of a signal is often insufficient. It is often more informative to consider how the signal would appear as a function of frequency, in the **frequency domain**.
2. Any time varying signal can be **decomposed into sinusoidal constituent components** of specific frequencies, phases, and amplitudes, just like the tidal level. This is the main idea of Fourier.
3. Two sinusoidal signals of different frequencies are **orthogonal** to each other, meaning that they have nothing in common, and it is not possible to “produce” one from the other through any linear methods.